

Supplement to “Dynamic project selection”

(Theoretical Economics, Vol. 13, No. 1, January 2018, 115–143)

ARINA NIKANDROVA

Department of Economics, Mathematics, and Statistics, Birkbeck, University of London

ROMANS PANCS

Centro de Investigación Económica, Instituto Tecnológico Autónomo de México

APPENDIX: AUXILIARY TECHNICAL LEMMAS

LEMMA A.1. (i) When $c < \underline{c}$, $\Phi^C(\cdot, c)$ on $[\underline{x}, \bar{x}]$ is positive at first, then intersects zero at a point, then is negative, then intersects zero at a point, and then is positive again.

(ii) The $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$ if and only if $c \geq \underline{c}$.

(iii) The $\Phi^A(\cdot, \cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]^2$ if and only if $c \geq \underline{c}$.

(iv) When $c < \bar{c}$, $\Phi^A(x_1, \cdot, c)$ is quasi-convex on $[x_1, b^{-1}(x_1)]$.

PROOF. The proof proceeds in steps.

Step 1. *Claim.* We have $1 - c - a > 0$ and $1 - c - a^2 > 0$ for any $a \in [\underline{x}, \bar{x}]$.

PROOF. We have

$$1 - c - a > 1 - c - \bar{x} = \frac{1 - 2c - \sqrt{1 - 4c}}{2} = \frac{4c^2}{2(1 - 2c + \sqrt{1 - 4c})} > 0,$$

which, coupled with $a^2 < a$, also implies that $1 - c - a^2 > 0$. $\triangleleft$

Step 2. *Claim.* Define $\hat{a} = 1 - \sqrt{c}$. Then $\hat{a} \in (\underline{x}, \bar{x})$, and $a \in [\underline{x}, \hat{a}] \cup (\hat{a}, \bar{x}]$ implies that $(a - \hat{a})(c - (1 - a)^2) > 0$.

PROOF. The inequality follows by the definition of $\hat{a}$ and by inspection. It remains to verify that $\hat{a} \in (\underline{x}, \bar{x})$. Indeed,

$$\begin{aligned}\hat{a} - \underline{x} &= \frac{1 + \sqrt{1 - 4c} - 2\sqrt{c}}{2} > 0, \\ \bar{x} - \hat{a} &= \frac{2\sqrt{c} + \sqrt{1 - 4c} - 1}{2} > 0,\end{aligned}$$

Arina Nikandrova: a.nikandrova@gmail.com

Romans Páncs: rpancs@gmail.com

where $c < \bar{c}$. $\triangleleft$

Step 3. Claim. If $\Phi^C(a, c) < 0$ for some $a \in [\underline{x}, \bar{x}]$, then $\Phi^C(\cdot, c)$ is at first positive, then intersects zero at a single point to the left of $\hat{a}$, then is negative, then intersects zero at a single point to the right of $\hat{a}$, and then is again positive.

PROOF. Because $\Phi^C(\underline{x}, c) = 1/(1 - \underline{x}) > 0$ and $\Phi^C(\bar{x}, c) = 1/(1 - \bar{x}) > 0$, then $\Phi(a, c) \leq 0 \implies a \in (\underline{x}, \bar{x})$. Differentiating yields

$$\begin{aligned}\Phi_1^C(a, c) &= \frac{c(1 - c - a^2)}{(1 - a)(1 - c - a)^2} + \ln \frac{c(1 - a)}{a(1 - c - a)} \\ &= \frac{\Phi^C(a, c)}{a} + \frac{(c - (1 - a)^2)(1 - c - a^2)}{a(1 - a)(1 - a - c)^2}.\end{aligned}$$

If $\Phi^C(a, c) = 0$ for some $a \in (\underline{x}, \bar{x})$, then

$$\Phi_1^C(a, c) = \frac{(c - (1 - a)^2)(1 - c - a^2)}{a(1 - a)(1 - c - a)^2}.$$

By Step 1, the sign of $\Phi_1^C(a, c)$ is the sign of $c - (1 - a)^2$, which, by Step 2, switches the sign from negative to positive at $\hat{a} \in (\underline{x}, \bar{x})$. Hence, if a with $\Phi^C(a, c) < 0$ exists, then $\Phi^C(\cdot, c)$ intersects zero twice: once from above and to the left of $\hat{a}$, and once from below and to the right of $\hat{a}$. $\triangleleft$

Step 4. Claim. If $c < \underline{c}$, then $\Phi^C(\hat{a}, c) < 0$; if $c \geq \underline{c}$, then $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$.

PROOF. Note that, at $a \in (\underline{x}, \bar{x})$,

$$\Phi_2^C(a, c) = \frac{a}{c} \left(\frac{1 - a}{1 - c - a} \right)^2 > 0.$$

Furthermore,

$$\Phi^C(\hat{a}, \underline{c}) = 2 - (1 - \sqrt{\underline{c}}) \ln \frac{(1 - \sqrt{\underline{c}})^2}{\underline{c}} = 0,$$

where the first equality is by $\hat{a} = 1 - \sqrt{\underline{c}}$ and the second equality is by (8). Combining the two displays above delivers $\Phi^C(\hat{a}, c) < 0$ for any $c < \underline{c}$ and $\Phi^C(\hat{a}, c) \geq 0$ for any $c \geq \underline{c}$.

For $c < \underline{c}$, $\Phi^C(\hat{a}, c) < 0$ and Step 3 imply part (i).

For $c \geq \underline{c}$, $\Phi^C(\hat{a}, c) \geq 0$ and Step 3 imply part (ii). $\triangleleft$

Step 5. Claim. Subject to $x_2 \geq x_1$, $\Phi^A(\cdot, \cdot, c)$ is minimized at $x_2 = x_1$.

PROOF. The claim follows from

$$\Phi_2^A(x_1, x_2, c) = \frac{(1 - x_1)}{(1 - x_2)^2} \left(\ln \frac{c(1 - x_1)}{x_1(1 - c - x_2)} + \frac{(1 - x_2)(1 - c - x_2^2)}{(1 - c - x_2)^2} \right) > 0,$$

where the inequality follows because $1 - c - x_2^2 > 0$ and $1 - c - x_2 > 0$ by Step 1.

Because $\Phi^A(z, z, c) = \Phi^C(z, c)$, Step 5 implies that $\Phi^A(\cdot, \cdot, c)$ has the same minimized value as $\Phi^C(\cdot, c)$ does. Hence, part (iii) is implied by part (ii). $\triangleleft$

Step 6. *Claim.* Define $\kappa(x_1, x_2, c) \equiv (1 - x_2)(2c - 1 + x_2 + x_2^2 - x_1 x_2^2) - c^2$. Then, for some $y^A \in (\underline{x}, \bar{x})$ and for any $x_2 \in [\underline{x}, y^*) \cup (y^*, \bar{x}]$, $(x_2 - y^A)\kappa(x_1, x_2, c) > 0$.

PROOF. First, we show that $\kappa(x_1, \bar{x}, c) > 0$. Indeed,

$$\kappa(x_1, \bar{x}, c) \geq \kappa(\bar{x}, \bar{x}, c) = c - (1 - \sqrt{1 - 4c})\left(\frac{1}{2} - c\right) > 0,$$

where the first inequality is by $\partial\kappa/\partial x_1 < 0$, and the last inequality follows because $\kappa(\bar{x}, \bar{x}, \cdot)$ is zero at $c \in \{0, \bar{c}\}$ and is positive at the only critical point ($c = 2/9$) in $(0, \bar{c})$.

Next, we show that $\kappa(x_1, \underline{x}, c) < 0$. Indeed,

$$\kappa(x_1, \underline{x}, c) \leq \kappa(\underline{x}, \underline{x}, c) = c - (1 + \sqrt{1 - 4c})\left(\frac{1}{2} - c\right) < 0,$$

where the first inequality is by $\partial\kappa/\partial x_1 < 0$, and the last inequality follows because $\kappa(\underline{x}, \underline{x}, \bar{c}) = 0$ and because $\partial\kappa(\underline{x}, \underline{x}, c)/\partial c > 0$.

Finally, $\partial^2\kappa(x_1, x_2, c)/x_2^2 = -6(1 - x_1)x_2 - 2x_1 < 0$. Hence, $\kappa(x_1, \underline{x}, c) < 0$ and $\kappa(x_1, \bar{x}, c) > 0$ imply that, on $(\underline{x}, \bar{x})$, $\kappa(x_1, \cdot, c)$ crosses zero and—by $\partial^2\kappa(x_1, x_2, c)/x_2^2 < 0$ —just once, from below, at some $y^A \in (\underline{x}, \bar{x})$. $\triangleleft$

Step 7. *Claim.* The $\Phi^A(x_1, \cdot, c)$ can be negative on and only on an interval.

PROOF. At any (x_1, x_2, c) with $\Phi^A(x_1, x_2, c) = 0$, by differentiation and substitution,

$$\Phi_2^A(x_1, x_2, x) = \frac{\kappa(x_1, x_2, c)}{(1 - x_2)x_2(1 - c - x_2)^2}.$$

The sign of $\Phi_2^A(x_1, x_2, c)$ is the sign of $\kappa(x_1, x_2, c)$, which, by Step 6, switches from negative to positive at $y^A \in (\underline{x}, \bar{x})$ as x_2 rises; $\Phi^A(x_1, x_2, c)$ is quasi-convex. Part (iv) follows. $\square$

LEMMA A.2. *The function M^C is uniquely maximized on $[\underline{x}, \bar{x}]$ at $\bar{x}$.*

PROOF. Recall from the proof of Lemma 3 that M^C has two local maxima: at $\underline{a}$ and at $\bar{x}$. It remains to verify that $M^C(\bar{x}) > M^C(\underline{a})$.

Then

$$M^C(\bar{x}) = 2c\sigma(\bar{x}) - \frac{1 - V(\bar{x}, \bar{x})}{(1 - \bar{x})^2} = 2c\sigma(\bar{x}) - \frac{1}{1 - \bar{x}},$$

where the last equality uses $V(\bar{x}, \bar{x}) = \bar{x}$, by direct substitution. Furthermore,

$$M^C(\underline{a}) = 2c\sigma(\underline{a}) - \frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2} = 2c\sigma(\underline{a}) - \frac{c}{\underline{a}(1 - \underline{a})^2} - \frac{1 - \underline{a}}{1 - \underline{a} - c},$$

where the last equality follows by substituting $\Phi(\underline{a}, c) = 0$ into the expression for $\frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2}$.

As a result,

$$\begin{aligned} M^C(\bar{x}) - M^C(\underline{a}) &= 2c\sigma(\bar{x}) - 2c\sigma(\underline{a}) + \frac{c}{\underline{a}(1 - \underline{a})^2} + \frac{1 - \underline{a}}{1 - \underline{a} - c} - \frac{1}{1 - \bar{x}} \\ &= c \left[2\sigma(\bar{x}) - \frac{1}{\bar{x}(1 - \bar{x})^2} \right] - c \left[2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2} \right] + \frac{1 - \underline{a}}{1 - \underline{a} - c} > 0, \end{aligned}$$

where the last equality follows from $\bar{x}(1 - \bar{x}) = c$ and by rearranging, and the inequality follows because the first bracket exceeds the second bracket, and the fraction $(1 - \underline{a})/(1 - \underline{a} - c)$ is positive (by $\underline{a} < \bar{x}$). The ordering of the brackets follows from $\bar{x} > \underline{a}$ and the observation

$$\frac{d}{da} \left(2\sigma(a) - \frac{1}{a(1 - a)} \right) = \frac{1}{a^2(1 - a)^2} > 0, \quad \forall a \in (0, 1).$$

To summarize, $M^C(\bar{x}) > M^C(\underline{a})$ and, so, M^C has a unique maximand, $\bar{x}$, on $[\underline{x}, \bar{x}]$. $\square$

LEMMA A.3. For M^A defined in (18), $\arg \max_{a \in [x_1, b^{-1}(x_1)]} M^A(x_1, a) = \{b^{-1}(x_1)\}$, where b^{-1} is the inverse of b defined in (11). As a result, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$.

PROOF. By Lemma 4, the only two local maxima of $M^A(x_1, \cdot)$ are $d(x_1)$ and $b^{-1}(x_1)$, so it suffices to show that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$. Write

$$M^A(x_1, b^{-1}(x_1)) = c\eta(b^{-1}(x_1)) - \frac{1 - V(x_1, b^{-1}(x_1))}{1 - b^{-1}(x_1)} = c\eta(b^{-1}(x_1)) - 1,$$

where the first equality is definitional and the second equality is by $V(x_1, b^{-1}(x_1)) = b^{-1}(x_1)$.

Evaluating V in (10) at $(x_1, d(x_1))$ and using $\Phi^A(x_1, d(x_1), c) = 0$ (by (19)), one can write

$$V(x_1, d(x_1)) = (1 - x_1)d(x_1) - \frac{(1 - d(x_1))c(1 - c - x_1d(x_1))}{d(x_1)(1 - c - d(x_1))} - c + x_1.$$

Then

$$\begin{aligned} M^A(x_1, d(x_1)) &= c\eta(d(x_1)) - \frac{1 - V(x_1, d(x_1))}{1 - d(x_1)} \\ &= c \left[\eta(d(x_1)) - \frac{1}{d(x_1)(1 - d(x_1))} \right] + x_1 - \frac{c(1 - x_1)}{1 - c - d(x_1)} - 1, \end{aligned}$$

where the first equality is definitional and the second equality follows by substituting $V(x_1, d(x_1))$ and rearranging.

Then, suppressing the argument x_1 in $b^{-1}(x_1)$ and in $d(x_1)$, for compactness,

$$\begin{aligned} M^A(x_1, b^{-1}) - M^A(x_1, d) &= c \left[\eta(b^{-1}) - \frac{1}{b^{-1}(1 - b^{-1})} - \left(\eta(d) - \frac{1}{d(1 - d)} \right) \right] \\ &\quad + \frac{b(d)(b^{-1} - x_1) + x_1(1 - b^{-1})}{b^{-1}(1 - b(d))} > 0, \end{aligned}$$

where the first equality follows by using the definitions of b and b^{-1} and rearranging, and the inequality uses $b^{-1}(x_1) > d(x_1)$ and

$$\frac{d}{dy} \left(\eta(y) - \frac{1}{y(1 - y)} \right) = \frac{1}{y^2(1 - y)} > 0$$

to conclude that the bracket in the first line is positive; and uses $x_1 < b^{-1}(x_1) < 1$ to conclude that the fraction in the second line is positive too. That is, $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$, as desired.

The conclusion that, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$ follows by inspection of Lemma's 4 Figure 7 (just validated by showing that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$). $\square$

LEMMA A.4. On $\hat{\mathcal{B}}$, $w(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{B}$.

PROOF. To conclude that $w(x_1) < b^{-1}(x_1)$, we show that $V(x_1, b^{-1}(x_1)) > B(x_1, b^{-1}(x_1))$.

Note that $V(x_1, b^{-1}(x_1)) = b^{-1}(x_1)$ and, from the definition of B in (22),

$$B(x_1, b^{-1}(x_1)) = 1 - (1 - b^{-1}(x_1)) \left(\frac{1 - C(x_1)}{1 - x_1} + c\eta(b^{-1}(x_1)) - c\eta(x_1) \right).$$

Then

$$V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1)) = (1 - b^{-1}(x_1)) \left(\frac{1 - C(x_1)}{1 - x_1} - 1 + c\eta(b^{-1}(x_1)) - c\eta(x_1) \right),$$

where

$$\begin{aligned} \frac{1 - C(x_1)}{1 - x_1} &= (1 - x_1) \left(\frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})] \right) \\ &= (1 - x_1) \left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})] \right). \end{aligned}$$

The first equality in the display above uses the definition of C in (13). The second equality uses the definitions of V in (10) and b in (11), and the condition $\Phi^C(\underline{a}, c) = 0$ in (15), which characterizes $\underline{a}$.

Then, substituting the display above into its precursor display gives

$$\begin{aligned}
& \frac{V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1))}{1 - b^{-1}(x_1)} \\
&= (1 - x_1) \left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})] \right) \\
&\quad + c[\eta(b^{-1}(x_1)) - \eta(x_1)] - 1 \\
&= \left(\frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - 1 \right. \\
&\quad \left. + (1 - x_1)c \left[2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2} \right) \right] \right. \\
&\quad \left. + c \left[\eta(b^{-1}(x_1)) - \frac{1}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)} \right) \right] \right).
\end{aligned}$$

Note that, using the definition of b in (11),

$$\begin{aligned}
& \frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - 1 \\
&= \frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{x_1}{b^{-1}(x_1)} - 1 \\
&= \frac{x_1[1 - b^{-1}(x_1)] + b(\underline{a})[b^{-1}(x_1) - x_1]}{b^{-1}(x_1)(1 - b(\underline{a}))} > 0,
\end{aligned}$$

where the inequality follows from $x_1 < b^{-1}(x_1) < 1$. Moreover,

$$2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2} \right) > 0$$

by $x_1 > \underline{a}$ and by

$$\frac{d}{dy} \left(2\sigma(y) - \frac{1}{y(1 - y)^2} \right) = \frac{1}{y^2(1 - y)^2} > 0$$

for any $y \in (0, 1)$. Finally,

$$\eta(b^{-1}(x_1)) - \frac{1}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)} \right) > 0$$

by $b^{-1}(x_1) > x_1$ and by

$$\frac{d}{dy} \left(\eta(y) - \frac{1}{y(1 - y)} \right) = \frac{1}{y^2(1 - y)} > 0$$

for $y \in (0, 1)$. Thus, $V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1)) > 0$, as required.

To show that $(\mathcal{F} \cap \hat{\mathcal{B}}) \subset \mathcal{B}$, from (22) and (10), write

$$\begin{aligned} \frac{B(x) - V(x)}{1 - x_2} &= (1 - x_1) \left(\frac{1}{1 - b(x_2)} + \frac{c}{1 - x_2} [\eta(x_1) - \eta(b(x_2))] \right) \\ &\quad - \frac{1 - C(x_1)}{1 - x_1} - c[\eta(x_2) - \eta(x_1)]. \end{aligned}$$

Differentiating and then simplifying gives

$$\frac{d}{dx_2} \left(\frac{B(x) - V(x)}{1 - x_2} \right) = - \frac{c\Phi^A(x, c)}{x_2(1 - x_2)}.$$

As a result, because $\Phi^A(x, c) < 0$ implies that $B(x) > V(x)$, $\mathcal{B}$ covers $\mathcal{F}$, the failure region, on $\hat{\mathcal{B}}$. That is, $\mathcal{F} \cap \hat{\mathcal{B}} \subset \mathcal{B}$. $\square$

Co-editor Johannes Hörner handled this manuscript.

Manuscript received 9 January, 2016; final version accepted 23 April, 2017; available online 1 May, 2017.